Systematic Derivation of Variational Expressions for Electromagnetic and/or Acoustic Waves

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***Abstract*—A systematic procedure generalized to derive the variational expressions for electromagnetic and/or acoustic field problems is proposed. It is shown that the variational expressions useful to treat the systems involving electromagnetic waves or acoustic waves or both can be formulated systematically all from the simple and basic principle of least action point of view.**

I. Introduction

A WIDESPREAD variety of miniaturized microwave circuits and functional devices can be realized by means of the coupling between electromagnetic waves and acoustic waves. However, the theoretical analysis of those circuits and devices is not easy, since both Maxwell's equations and Newton's equation must be solved simultaneously. Therefore, the suitable approximate methods of analysis are desired. The variational method is one of the powerful techniques to treat such problems.

In applying the variational method, the most important and difficult problem is to find the appropriate variational expressions. The authors have proposed previously the unified procedure to derive the variational expressions for the electromagnetic field problems from the principle of least action point of view [1]. In the present paper, not only the electromagnetic fields but also the acoustic fields are taken into account, thereby enabling one to derive the various variational expressions for the electromagnetic and/or acoustic field problems systematically from the same principle. To make clear the point of an argument, discussions are limited to the system in which the coupling occurs between the electric fields and the acoustic fields, i.e., the electroacoustic wave systems. However, the method given in this paper is applicable to derive the variational expressions for the magnetoacoustic wave systems as well. As typical examples, the variational expressions for a resonant frequency and a propagation constant of the electroacoustic wave systems are derived systematically all from the least action principle.

II. The Principle of Least Action

The action J for the system which contains both electromagnetic waves and acoustic waves consists of three terms as follows:

$$J = J_f + J_{fm} + J_m \tag{1}$$

where J_f, J_m, and J_{fm} represent, respectively, the terms relating to the electromagnetic field, the acoustic field and the coupling between them. Let us assume that the materials involved are linear but inhomogeneous and anisotropic in general. The action J_f is then expressed as [1]

$$J_f = \int_{t_0}^{t_1} dt \int_V \Big(\frac{1}{2}\{ \boldsymbol{E}(\boldsymbol{r},t)\cdot\hat{\varepsilon}(\boldsymbol{r},t)\cdot\boldsymbol{E}(\boldsymbol{r},t) - \boldsymbol{H}(\boldsymbol{r},t)\cdot\hat{\mu}(\boldsymbol{r},t)\cdot\boldsymbol{H}(\boldsymbol{r},t)\} + \boldsymbol{A}(\boldsymbol{r},t)\cdot\boldsymbol{J}(\boldsymbol{r},t) - \rho(\boldsymbol{r},t)\phi(\boldsymbol{r},t)\Big)\, dv \tag{2}$$

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} - \nabla\phi(\boldsymbol{r},t) \tag{3}$$

where $\boldsymbol{r}$ and t are the vector distance from the origin and the time, $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and the magnetic field intensities, $\boldsymbol{A}$ and ϕ are the vector and the scalar potentials, $\boldsymbol{J}$ and ρ are the electric current density and the electric charge density, and $\hat{\varepsilon}$ and $\hat{\mu}$ are the tensor permittivity and the tensor permeability, respectively. t_0 is the initial time and t_1 is the final time.

The action J_m relating to the acoustic field is defined as

$$J_m = \int_{t_0}^{t_1} dt \int_V \Big(\frac{1}{2} m(\boldsymbol{r}) \Big\{ \frac{\partial \boldsymbol{u}(\boldsymbol{r},t)}{\partial t} \Big\}^2 - \frac{1}{2}\boldsymbol{S}(\boldsymbol{r},t):\boldsymbol{c}(\boldsymbol{r},t):\boldsymbol{S}(\boldsymbol{r},t) + \boldsymbol{F}(\boldsymbol{r},t)\cdot\boldsymbol{u}(\boldsymbol{r},t)\Big)\, dv \tag{4}$$

$$\boldsymbol{S}(\boldsymbol{r},t) = \frac{1}{2}\Big\{ \nabla\boldsymbol{u}(\boldsymbol{r},t) + \widetilde{(\nabla\boldsymbol{u}(\boldsymbol{r},t))} \Big\} \equiv \nabla_s \boldsymbol{u}(\boldsymbol{r},t) \tag{5}$$

where m and $\boldsymbol{u}$ are the mass density of the medium and the particle displacement field, $\boldsymbol{S}$ is the strain field, $\boldsymbol{F}$ and $\boldsymbol{c}$ are the body force and the elastic stiffness constant, respectively, and the tilde $\sim$ designates a transposed tensor.

Limiting our discussions to the electroacoustic wave systems, the term J_{fm} relating to the coupling between the electromagnetic field and the acoustic field is given as

$$J_{fm} = \int_{t_0}^{t_1} dt \int_V \Big(\frac{1}{2}\{ \boldsymbol{E}(\boldsymbol{r},t)\cdot\boldsymbol{e}(\boldsymbol{r},t):\boldsymbol{S}(\boldsymbol{r},t) + \boldsymbol{S}(\boldsymbol{r},t):\underset{\sim}{\boldsymbol{e}}(\boldsymbol{r},t)\cdot\boldsymbol{E}(\boldsymbol{r},t)\}\Big)\, dv \tag{6}$$

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where e and $\underline{e}$ are the piezoelectric stress constants. Using the foregoing equations, (1) is reduced to

$$J=\int_{t_0}^{t_1} dt \int_V \left(\frac{1}{2}(E\cdot D - H\cdot B) + A\cdot J - \rho\phi + \frac{1}{2} m \left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2} S:T + F\cdot u \right) dv \quad (7)$$

where

$$D(r,t)=\hat{\varepsilon}(r,t)\cdot E(r,t)+e(r,t):S(r,t),$$
$$T(r,t)=-\underline{e}(r,t)\cdot E(r,t)+c(r,t):S(r,t),$$
$$B(r,t)=\hat{\mu}(r,t)\cdot H(r,t) \quad (8)$$

and T is the stress field. In particular, in the source free case ($J=0$, $\rho=0$, $F=0$), (7) is reduced to the following form by using the quasistatic approximation ($E=-\nabla\phi$):

$$J=\int_{t_0}^{t_1} dt \int_V \left(\frac{1}{2} E\cdot D + \frac{1}{2} m \left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2} S:T \right) dv. \quad (9)$$

This expression coincides with the equation given by Auld [2]. In other words, (7) contains Auld's equation (9) as its special case.

Let us consider next another expression of the principle of least action for the electroacoustic field in the frequency domain. Extending the time interval $[t_0,t_1]$ to $(-\infty,\infty)$ and applying Fourier transformation, the integration with respect to time t in (7) becomes the integration with respect to the frequency f, and thus the time domain problem of the electroacoustic fields can be transformed into the frequency domain problem. Assuming that the materials involved are dissipationless, and have neither permanent electric polarization nor permanent magnetic polarization, the components of the tensor material constants have the following relations:

$$\varepsilon_{ij}=\varepsilon_{ji}^*, \qquad \mu_{ij}=\mu_{ji}^*$$
$$e_{ijk}=\underline{e}_{jki}^*, \qquad c_{ijkl}=c_{klij}^* \quad (10)$$

and the stress matrix T is symmetric. Making use of (10), the time domain expression (7) becomes the frequency domain expression in the form

$$J=\int_0^\infty df \int_V (E^*\cdot\hat{\varepsilon}\cdot E - H^*\cdot\hat{\mu}\cdot H + A\cdot J^* + A^*\cdot J - \rho\phi^* - \rho^*\phi + S^*:\underline{e}\cdot E + S:\underline{e}^*\cdot E^* + m\omega^2 u^*\cdot u - S^*:c:S + F\cdot u^* + F^*\cdot u)\, dv. \quad (11)$$

The first-order variations of the action J due to the small variations in ϕ, A, and u are derived, respectively, as

$$\delta J_\phi = \int_0^\infty df \int_V (\delta\phi^*(\nabla\cdot D - \rho))\, dv - \int_0^\infty df \int_{S+S_d} (\delta\phi^* D)\cdot n\, ds + cc \quad (12)$$

$$\delta J_A = \int_0^\infty df \int_V (\delta A^*\cdot(j\omega D + J - \nabla\times H))\, dv + \int_0^\infty df \int_{S+S_d} (H\times\delta A^*)\cdot n\, ds + cc \quad (13)$$

$$\delta J_u = \int_0^\infty df \int_V (\delta u^*\cdot(\nabla\cdot T + m\omega^2 u + F))\, dv - \int_0^\infty df \int_{S+S_d} (T\cdot\delta u^*)\cdot n\, ds + cc \quad (14)$$

where the terms designated as cc represent the complex conjugate of the first two terms on the right-hand sides of the preceding equations, S is the surface which encloses the volume V, and S_d is the discontinuous boundary of the materials involved. Therefore, in order to make J stationary, A, ϕ, and u in the region where the materials are continuous must satisfy the following:

$$\nabla\cdot D=\rho \quad (15)$$
$$\nabla\times H=j\omega D+J \quad (16)$$
$$\nabla\cdot T=-m\omega^2 u - F. \quad (17)$$

Further, from the frequency domain expression of (3), the following is yielded:

$$\nabla\times E=-j\omega B \quad (18)$$

where

$$B=\nabla\times A. \quad (19)$$

Equations (15)–(18) represent the Maxwell's equations and the Newton's equation. It should be pointed out, therefore, that if A, ϕ, and u are determined in such a manner that for those A, ϕ, and u the action J becomes stationary, the Maxwell's equations, and the Newton's equation are satisfied. In other words, the physically realizable electromagnetic and acoustic fields satisfying the Maxwell's equations and the Newton's equation can be derived from the least action principle.

III. Variational Expressions for Resonant Frequency

In this section, we shall derive the variational expressions for the resonant frequency of the electroacoustic resonating systems from the action J obtained in the preceding section. We will assume that the resonating system is composed of the linear, nondispersive, and dissipationless piezoelectric materials. The resonating system is assumed to be source free, i.e., $J=0$, $\rho=0$, and $F=0$ in the region V (Fig. 1). Then, the action J given by (11) is reduced to

$$J=\int_0^\infty df \int_V (E^*\cdot\hat{\varepsilon}\cdot E - H^*\cdot\hat{\mu}\cdot H + S^*:\underline{e}\cdot E + S:\underline{e}^*\cdot E^* + m\omega^2 u^*\cdot u - S^*:c:S)\, dv \quad (20)$$

where V is the region within the resonator. In (20), independent variables are A, ϕ, and u. Let us transform A, ϕ,

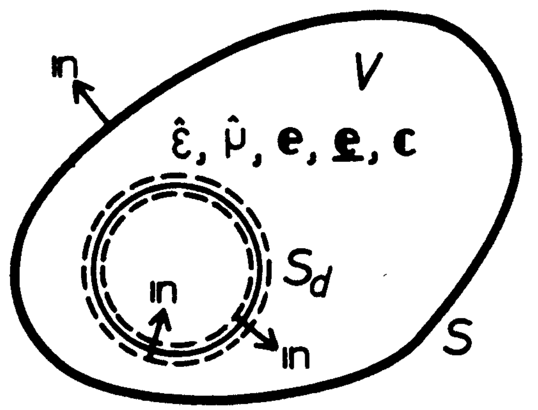

Fig. 1. Resonator. Volume V enclosed by surface S. S_d indicates the surface across which materials involved change discontinuously. $\boldsymbol{n}$ is a unit vector normal to the boundary surface.

and $\boldsymbol{u}$ into $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{v}$, and $\boldsymbol{S}$ by means of the following:

$$\begin{aligned} \boldsymbol{E} &= -j\omega \boldsymbol{A} - \nabla\phi \\ \boldsymbol{H} &= \hat{\mu}^{-1}\cdot \nabla \times \boldsymbol{A} \\ \boldsymbol{v} &= j\omega \boldsymbol{u} \\ \boldsymbol{S} &= \nabla_s \boldsymbol{u} \end{aligned} \tag{21}$$

where $\boldsymbol{v}$ is the particle velocity. Further, substituting the following identity

$$\nabla_s \boldsymbol{v}^* : \boldsymbol{T} = \nabla\cdot(\boldsymbol{T}\cdot\boldsymbol{v}^*) - \boldsymbol{v}^*\cdot(\nabla\cdot\boldsymbol{T}) \tag{22}$$

into (20) and applying the Gauss' theorem, we get the action for J in the form

$$\begin{aligned} J = \int_0^\infty df \int_V \Big(& \boldsymbol{E}^*\cdot\hat{\varepsilon}\cdot\boldsymbol{E} + \boldsymbol{H}^*\cdot\hat{\mu}\cdot\boldsymbol{H} + m\boldsymbol{v}^*\cdot\boldsymbol{v} + \boldsymbol{S}^* : \boldsymbol{c} : \boldsymbol{S} \\ & - \frac{j}{\omega}\{\boldsymbol{H}^*\cdot\nabla\times\boldsymbol{E} - \boldsymbol{E}^*\cdot\nabla\times\boldsymbol{H} - \boldsymbol{v}^*\cdot(\nabla\cdot\boldsymbol{T}) - \nabla_s\boldsymbol{v} : \boldsymbol{T}^*\}\Big) dv \\ & - \int_0^\infty df \int_{S+S_d} \left(\frac{j}{\omega}\{\boldsymbol{H}\times\boldsymbol{E}^* + \boldsymbol{T}\cdot\boldsymbol{v}^*\}\right)\cdot \boldsymbol{n}\, ds. \end{aligned} \tag{23}$$

Since the electroacoustic field in a resonator can be expressed by a linear combination of the electroacoustic fields of the individual resonant modes, the field quantities, $\boldsymbol{E}(\boldsymbol{r},\omega)$ for instance, can be written in the form

$$\boldsymbol{E}(\boldsymbol{r},\omega) = \sum_i \delta(\omega-\omega_i)\boldsymbol{E}_i(\boldsymbol{r}) \tag{24}$$

where $\boldsymbol{E}_i(\boldsymbol{r})$ and ω_i are the electric field and the resonant frequency of the ith mode, respectively, and $\delta(\omega-\omega_i)$ signifies the delta function. Substituting (24) into (23), carrying out the integration with respect to the frequency f, and assuming that there exists only ith mode alone at $t=-\infty$, we get

$$J = L\delta(0) \tag{25}$$

where

$$\begin{aligned} L = \int_V \Big(& \boldsymbol{E}^*\cdot\hat{\varepsilon}\cdot\boldsymbol{E} + \boldsymbol{H}^*\cdot\hat{\mu}\cdot\boldsymbol{H} + m\boldsymbol{v}^*\cdot\boldsymbol{v} + \boldsymbol{S}^* : \boldsymbol{c} : \boldsymbol{S} \\ & - \frac{j}{\omega}\{\boldsymbol{H}^*\cdot\nabla\times\boldsymbol{E} - \boldsymbol{E}^*\cdot\nabla\times\boldsymbol{H} - \boldsymbol{v}^*\cdot(\nabla\cdot\boldsymbol{T}) - \nabla_s\boldsymbol{v} : \boldsymbol{T}^*\}\Big) dv \\ & - \frac{j}{\omega}\int_{S+S_d} (\boldsymbol{H}\times\boldsymbol{E}^* + \boldsymbol{T}\cdot\boldsymbol{v}^*)\cdot\boldsymbol{n}\, ds. \end{aligned} \tag{26}$$

The subscript i will be omitted hereafter for simplicity. The stationary problem for J is thus reduced to that for L. Since L given by (26) is zero for the correct values of the field quantities $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{v}$, and $\boldsymbol{S}$ as shown in the Appendix, we obtain the variational expression for the resonant frequency of the electroacoustic resonating systems as follows:

$$\omega = \frac{j\int_V (\boldsymbol{H}^*\cdot\nabla\times\boldsymbol{E} - \boldsymbol{E}^*\cdot\nabla\times\boldsymbol{H} - \boldsymbol{v}^*\cdot(\nabla\cdot\boldsymbol{T}) - \nabla_s\boldsymbol{v} : \boldsymbol{T}^*)\, dv + j\int_{S+S_d} (\boldsymbol{H}\times\boldsymbol{E}^* + \boldsymbol{T}\cdot\boldsymbol{v}^*)\cdot\boldsymbol{n}\, ds}{\int_V (\boldsymbol{E}^*\cdot\hat{\varepsilon}\cdot\boldsymbol{E} + \boldsymbol{H}^*\cdot\hat{\mu}\cdot\boldsymbol{H} + m\boldsymbol{v}^*\cdot\boldsymbol{v} + \boldsymbol{S}^* : \boldsymbol{c} : \boldsymbol{S})\, dv} \tag{27}$$

where $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{v}$, and $\boldsymbol{S}$ are the functions of position vector $\boldsymbol{r}$, and $\boldsymbol{T}$ is given by

$$\boldsymbol{T}(\boldsymbol{r}) = -\underline{e}(\boldsymbol{r})\cdot\boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{c}(\boldsymbol{r}) : \boldsymbol{S}(\boldsymbol{r}). \tag{28}$$

The trial functions for $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{v}$, and $\boldsymbol{S}$ in (27) must satisfy the following conditions:

1) $\boldsymbol{n}\times\delta\boldsymbol{E}$ and $\delta\boldsymbol{v}$ are zero on S. Note that if S is a magnetic wall, the former condition ($\boldsymbol{n}\times\delta\boldsymbol{E}=0$ on S) is not required, and if S is a stress-free boundary, the later condition ($\delta\boldsymbol{v}=0$ on S) is not required.

2) $\boldsymbol{n}\times\delta\boldsymbol{E}$ and $\delta\boldsymbol{v}$ are continuous across S_d.

In the quasi-static case, $\boldsymbol{E}$ and $\boldsymbol{H}$ in (27) can be expressed as

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) &= -\nabla\phi(\boldsymbol{r}) \\ \boldsymbol{H}(\boldsymbol{r}) &= 0. \end{aligned} \tag{29}$$

Substituting (29) into (27), we get the variational expression for the resonant frequency of the electroacoustic resonating systems under the quasistatic approximation as follows:

$$\omega = \frac{-j\int_V (\boldsymbol{v}^*\cdot(\nabla\cdot\boldsymbol{T}) + \nabla_s\boldsymbol{v} : \boldsymbol{T}^*)\, dv + j\int_{S+S_d} (\boldsymbol{T}\cdot\boldsymbol{v}^*)\cdot\boldsymbol{n}\, ds}{\int_V (\nabla\phi^*\cdot\hat{\varepsilon}\cdot\nabla\phi + m\boldsymbol{v}^*\cdot\boldsymbol{v} + \boldsymbol{S}^* : \boldsymbol{c} : \boldsymbol{S})\, dv} \tag{30}$$

where

$$\boldsymbol{T}(\boldsymbol{r}) = \underline{e}(\boldsymbol{r})\cdot\nabla\phi(\boldsymbol{r}) + \boldsymbol{c}(\boldsymbol{r}) : \boldsymbol{S}(\boldsymbol{r}). \tag{31}$$

From the first-order variation of ω given by (30), the conditions which must be satisfied by the trial functions for ϕ, $\boldsymbol{v}$, and $\boldsymbol{S}$ can be obtained as follows:

1) $\delta\phi$ and $\delta\boldsymbol{v}$ are zero on S. Note that if S is a stress-free boundary, the later condition ($\delta\boldsymbol{v}=0$ on S) is not required.

2) $\delta\phi$ and $\delta\boldsymbol{v}$ are continuous across S_d.

Further, by setting $\boldsymbol{E}=\boldsymbol{H}=0$ in (27), we obtain a variational expression for the resonant frequency of the acoustic wave resonator. On the contrary, if we set $\boldsymbol{v}=\boldsymbol{S}=0$, (27) becomes a variational expression for the resonant frequency of the electromagnetic wave resonator which coincides with the result obtained previously [1].

IV. Variational Expressions for Propagation Constant

Let us derive next the variational expressions for the propagation constant of the electroacoustic waves traveling along a uniform wave guiding system. It is assumed that the materials involved are inhomogeneous and anisotropic, in general, but are linear, nondispersive, and dissipation free. Fig. 2 illustrates the wave guiding system under consideration which is made of the piezoelectric material and is uniform in a direction of wave propagation z.

To derive the variational expressions for the propagation constant, we divide the volume integral in (26) into the surface integral over the transverse plane S(xy plane) and the integral along the propagation axis z. Further, transforming the integration with respect to z into the integration with respect to the propagation constant β by performing Fourier transformation, the following equation is obtained.

$$\begin{aligned} L=\frac{1}{2\pi\omega}\int_{-\infty}^{\infty} d\beta\Bigg(\int_S &(\omega \boldsymbol{E}(x,y,\beta)^*\cdot\hat{\varepsilon}(x,y)\\ &\cdot\boldsymbol{E}(x,y,\beta)+\omega\boldsymbol{H}(x,y,\beta)^*\cdot\hat{\mu}(x,y)\cdot\boldsymbol{H}(x,y,\beta)\\ &+\omega m(x,y)\boldsymbol{v}(x,y,\beta)^*\cdot\boldsymbol{v}(x,y,\beta)\\ &+\omega\boldsymbol{S}(x,y,\beta)^*:\boldsymbol{c}(x,y):\boldsymbol{S}(x,y,\beta)\\ &-j\{\boldsymbol{H}(x,y,\beta)^*\cdot\nabla_t\times\boldsymbol{E}(x,y,\beta)\\ &-\boldsymbol{E}(x,y,\beta)^*\cdot\nabla_t\times\boldsymbol{H}(x,y,\beta)\\ &-\boldsymbol{v}(x,y,\beta)^*\cdot\nabla_t\cdot\boldsymbol{T}(x,y,\beta)\\ &-\nabla_{ts}\boldsymbol{v}(x,y,\beta):\boldsymbol{T}(x,y,\beta)^*\}\\ &+\beta\boldsymbol{i}_z\cdot\{\boldsymbol{E}(x,y,\beta)\times\boldsymbol{H}(x,y,\beta)^*\\ &-\boldsymbol{H}(x,y,\beta)\times\boldsymbol{E}(x,y,\beta)^*\\ &-\boldsymbol{T}(x,y,\beta)\cdot\boldsymbol{v}(x,y,\beta)^*\\ &-\boldsymbol{T}(x,y,\beta)^*\cdot\boldsymbol{v}(x,y,\beta)\}) \,ds\\ &-j\int_{C+C_d}(\boldsymbol{H}(x,y,\beta)\times\boldsymbol{E}(x,y,\beta)^*\\ &+\boldsymbol{T}(x,y,\beta)\cdot\boldsymbol{v}(x,y,\beta)^*)\cdot\boldsymbol{n}\,dl\Bigg). \end{aligned} \tag{32}$$

In the foregoing equation, C represents a closed contour in the transverse plane as shown in Fig. 3. For the open-type wave guiding structure, C must be a closed contour enclosing the wave guiding structure at infinity, while in the case of the closed metallic waveguide, C is a cross-sectional boundary of the guide wall. C_d indicates the line in the transverse plane across which the material constants change discontinuously. $\boldsymbol{E}(x, y, \beta)$ and $\boldsymbol{H}(x,y,\beta)$ are the Fourier transforms of $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{H}(\boldsymbol{r})$, and ∇_t and ∇_{ts} indicate the transverse parts of the operator ∇ and ∇_s, respectively.

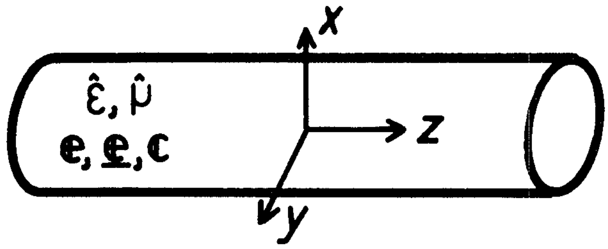

Fig. 2. Wave guiding structure containing inhomogeneous and anisotropic materials. Direction of wave propagation is in z axis along which the structure is uniform.

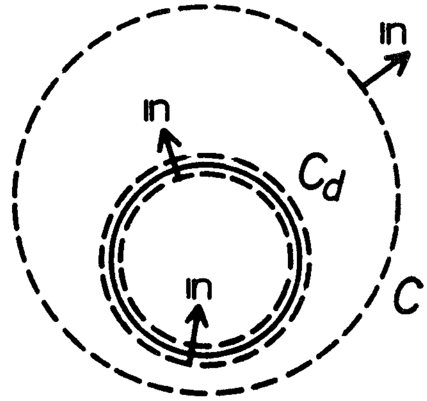

Fig. 3. Contours used in the evaluation of integration.

The electroacoustic fields of the wave propagating in the z direction can be expressed in terms of the linear combinations of the electroacoustic fields of each propagation mode. Hence the field quantities, $\boldsymbol{E}(x,y,\beta)$ for instance, can be written in the form

$$\boldsymbol{E}(x,y,\beta)=\sum_i \delta(\beta+\beta_i)\boldsymbol{E}_i(x,y) \tag{33}$$

where $\boldsymbol{E}_i(x,y)$ represents the electric field of the ith mode, and β_i is its propagation constant. Substituting (33) into (32), carrying out the integration with respect to the propagation constant β, and assuming that there exists only ith mode alone at $t=-\infty$, we get

$$J=\frac{\delta(0)}{2\pi\omega}M \tag{34}$$

where

$$\begin{aligned} M=\int_S&(\omega\boldsymbol{E}^*\cdot\hat{\varepsilon}\cdot\boldsymbol{E}+\omega\boldsymbol{H}^*\cdot\hat{\mu}\cdot\boldsymbol{H}+\omega m\boldsymbol{v}^*\cdot\boldsymbol{v}+\omega\boldsymbol{S}^*:\boldsymbol{c}:\boldsymbol{S}\\ &-j(\boldsymbol{H}^*\cdot\nabla_t\times\boldsymbol{E}-\boldsymbol{E}^*\cdot\nabla_t\times\boldsymbol{H}-\boldsymbol{v}^*\cdot\nabla_t\cdot\boldsymbol{T}-\nabla_{ts}\boldsymbol{v}:\boldsymbol{T}^*)\\ &-\beta\boldsymbol{i}_z\cdot(\boldsymbol{E}\times\boldsymbol{H}^*-\boldsymbol{H}\times\boldsymbol{E}^*-\boldsymbol{T}\cdot\boldsymbol{v}^*-\boldsymbol{T}^*\cdot\boldsymbol{v}))\,ds\\ &-j\int_{C+C_d}(\boldsymbol{H}\times\boldsymbol{E}^*+\boldsymbol{T}\cdot\boldsymbol{v}^*)\cdot\boldsymbol{n}\,dl. \end{aligned} \tag{35}$$

The subscript i has been omitted for simplicity. Thus the stationary problem for L is reduced to that for M, and hence we can determine the correct values of $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{v}$, $\boldsymbol{S}$, and β in such a way as M becomes stationary for the correct values of those quantities. M given by (35) vanishes for the correct values of $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{v}$, and $\boldsymbol{S}$ as shown in the Appendix. Therefore, by steps similar to those used in the preceding section, we get the variational expression for the propagation constant β as follows:

$$\beta=\frac{\int_S(\omega \boldsymbol{E}^*\cdot\hat{\varepsilon}\cdot\boldsymbol{E}+\omega\boldsymbol{H}^*\cdot\hat{\mu}\cdot\boldsymbol{H}+\omega m\boldsymbol{v}^*\cdot\boldsymbol{v}+\omega\boldsymbol{S}^*:\boldsymbol{c}:\boldsymbol{S}-j(\boldsymbol{H}^*\cdot\nabla_t\times\boldsymbol{E}-\boldsymbol{E}^*\cdot\nabla_t\times\boldsymbol{H}-\boldsymbol{v}^*\cdot\nabla_t\cdot\boldsymbol{T}-\nabla_{ts}\boldsymbol{v}:\boldsymbol{T}^*))\,ds-j\int_{C+C_d}(\boldsymbol{H}\times\boldsymbol{E}^*+\boldsymbol{T}\cdot\boldsymbol{v}^*)\cdot\boldsymbol{n}\,dl}{\int_S(\boldsymbol{i}_z\cdot(\boldsymbol{E}\times\boldsymbol{H}^*-\boldsymbol{H}\times\boldsymbol{E}^*-\boldsymbol{T}\cdot\boldsymbol{v}^*-\boldsymbol{T}^*\cdot\boldsymbol{v}))\,ds} \tag{36}$$

where the variables $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{v}$, and $\boldsymbol{S}$ in (36) are functions of the transverse coordinates x and y, and $\boldsymbol{T}$ is given by

$$\boldsymbol{T}(x,y)=-\underline{\boldsymbol{e}}(x,y)\cdot\boldsymbol{E}(x,y)+\boldsymbol{c}(x,y):\boldsymbol{S}(x,y). \tag{37}$$

By calculating the first variation of β, we get the conditions of the functions for $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{v}$, and $\boldsymbol{S}$ as follows:

1) $\boldsymbol{n}\times\delta\boldsymbol{E}$ and $\delta\boldsymbol{v}$ are zero along C. Note that if C is a magnetic wall, the former condition ($\boldsymbol{n}\times\delta\boldsymbol{E}=0$ along C) is not required, and if C is a stress-free boundary, the later condition ($\delta\boldsymbol{v}=0$ along C) is not required.

2) $\boldsymbol{n}\times\delta\boldsymbol{E}$ and $\delta\boldsymbol{v}$ are continuous across C_d.

By changing the line integral terms in (36) appropriately, the alternate variational expressions can be yielded [1]. The conditions to the trial functions are altered also from those stated above.

In the quasi-static case, $\boldsymbol{E}$ and $\boldsymbol{H}$ in (26) can be expressed in the form given by (29). According to the similar procedure as before, divide the volume integral into the surface integral over the transverse plane S(xy plane) and the integral along the propagation axis z, and transform the integration with respect to z into the integration with respect to the propagation constant β by performing Fourier transformation. Then, we get the following variational expression for a propagation constant under the quasi-static approximation.

$$\beta=\frac{\int_S\left(\boldsymbol{u}^*\cdot\nabla_t\cdot\boldsymbol{T}-\boldsymbol{T}^*:\nabla_{ts}\boldsymbol{u}+m\omega^2\boldsymbol{u}^*\cdot\boldsymbol{u}+\boldsymbol{S}:\boldsymbol{T}^*-\boldsymbol{E}\cdot\boldsymbol{D}^*-\nabla_t\phi\cdot\boldsymbol{D}^*-\nabla_t\phi^*\cdot\boldsymbol{D}\right)ds-\int_{C+C_d}(\boldsymbol{T}\cdot\boldsymbol{u}^*)\cdot\boldsymbol{n}\,dl}{j\int_S(\boldsymbol{i}_z\cdot(\boldsymbol{T}\cdot\boldsymbol{u}^*-\boldsymbol{T}^*\cdot\boldsymbol{u}+\phi^*\boldsymbol{D}-\phi\boldsymbol{D}^*))\,ds} \tag{38}$$

where

$$\boldsymbol{D}(x,y)=\hat{\varepsilon}(x,y)\cdot\boldsymbol{E}(x,y)+\boldsymbol{e}(x,y):\boldsymbol{S}(x,y)$$
$$\boldsymbol{T}(x,y)=-\underline{\boldsymbol{e}}(x,y)\cdot\boldsymbol{E}(x,y)+\boldsymbol{c}(x,y):\boldsymbol{S}(x,y). \tag{39}$$

The variational expression given by (38) coincides with that obtained by Makimoto *et al.* [3]. The conditions which must be satisfied by the trial functions are given as follows:

1) $\delta\boldsymbol{u}$ and $\delta\phi$ are zero along C. Note that if C is a stress-free boundary, the former condition ($\delta\boldsymbol{u}=0$ along C) is not required.

2) $\delta\boldsymbol{u}$ and $\delta\phi$ are continuous across C_d.

V. Conclusion

A systematic approach generalized to derive the variational expressions for not only electromagnetic field but also acoustic field problems has been proposed. It has been shown that various variational expressions for electromagnetic fields, acoustic fields, and electroacoustic fields can be derived systematically all from the least action principle. To make clear the point of an argument, the discussions have been limited to the system in which the coupling occurs between electric and acoustic fields. However, the method given in this paper is applicable to derive the variational expressions for the magnetoacoustic systems as well in which the magnetic and acoustic fields are coupled together. It has been shown that by determining $\boldsymbol{A}$, ϕ, and $\boldsymbol{u}$ in such a manner that the action J becomes stationary for those $\boldsymbol{A}$, ϕ, and $\boldsymbol{u}$, the associated electromagnetic and acoustic fields satisfy the Maxwell's equations and the Newton's equation. It follows that we can expect to derive the various variational expressions for electromagnetic fields, acoustic fields, electroacoustic fields, and magnetoacoustic fields all from "the principle of least action" point of view.

VI. Appendix

For the source free region V, (11) is reduced to

$$J=\int_0^\infty df\int_V\left(\frac{1}{2}(\boldsymbol{E}^*\cdot\boldsymbol{D}-\boldsymbol{H}\cdot\boldsymbol{B}^*)+\frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D}^*-\boldsymbol{H}^*\cdot\boldsymbol{B})+\frac{1}{2}(m\omega^2\boldsymbol{u}^*\cdot\boldsymbol{u}-\boldsymbol{S}:\boldsymbol{T}^*)+\frac{1}{2}(m\omega^2\boldsymbol{u}^*\cdot\boldsymbol{u}-\boldsymbol{S}^*:\boldsymbol{T})\right)dv. \tag{A1}$$

On the other hand, by assuming that electroacoustic fields satisfy the Maxwell's equations and the Newton's equation, the following equations are derived.

$$\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}^*)=j\omega(\boldsymbol{E}\cdot\boldsymbol{D}^*-\boldsymbol{H}^*\cdot\boldsymbol{B})$$
$$\nabla\cdot(-\boldsymbol{v}^*\cdot\boldsymbol{T})=-j\omega(m\omega^2\boldsymbol{u}^*\cdot\boldsymbol{u}-\boldsymbol{S}^*:\boldsymbol{T}). \tag{A2}$$

Substituting (A2) and its complex conjugation into (A1)

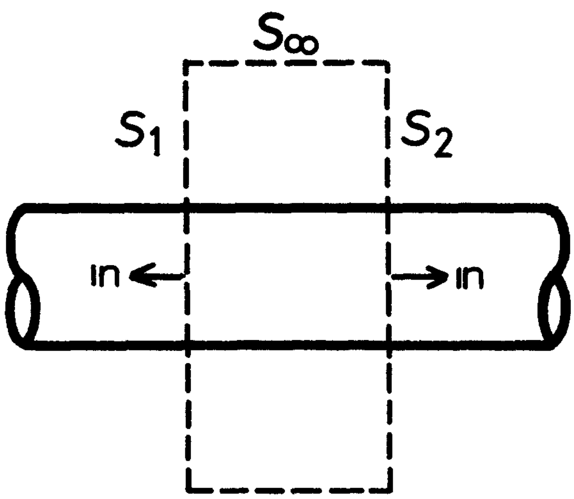

Fig. 4. Surface $S(=S_1+S_2+S_\infty)$ used in the evaluation of integration in (A3). S_∞ is a cylindrical side surface with infinite radius, and S_1 and S_2 are parallel surfaces transverse to the propagation direction of the wave guiding structure.

and applying Gauss' theorem, we get

$$J=\int_0^\infty df\int_{S+S_d} \cdot\left(\frac{1}{2j\omega}(\boldsymbol{E}\times\boldsymbol{H}^*-\boldsymbol{E}^*\times\boldsymbol{H}-\boldsymbol{v}\cdot\boldsymbol{T}^*+\boldsymbol{v}^*\cdot\boldsymbol{T})\right)\cdot\boldsymbol{n}\,ds. \quad \text{(A3)}$$

Since $\boldsymbol{n}\times\boldsymbol{E}$, $\boldsymbol{n}\times\boldsymbol{H}$, $\boldsymbol{v}$, and $\boldsymbol{T}\cdot\boldsymbol{n}$ must be continuous across the discontinuity surface S_d, the integration over S_d in (A3) is zero. $\boldsymbol{n}\times\boldsymbol{E}$ or $\boldsymbol{n}\times\boldsymbol{H}$ must be zero on the resonator surface S for electromagnetic fields, and $\boldsymbol{T}\cdot\boldsymbol{n}$ or $\boldsymbol{v}$ must be zero on S for acoustic fields. Hence $J=0$. As shown in Fig. 4, the surface S consists of S_1, S_2, and S_∞. J given by (A3) vanishes for a propagation mode because $(\boldsymbol{E}\times\boldsymbol{H}^*)\cdot\boldsymbol{n}$ and $(\boldsymbol{v}\cdot\boldsymbol{T}^*)\cdot\boldsymbol{n}$ are zero on S_∞ and also the unit normal vectors $\boldsymbol{n}$ on S_1 and S_2 direct to opposite directions. Therefore, we can conclude that the action J must be zero for the correct resonant modes and propagation modes.

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